Solitons and diffusive modes in the noiseless Burgers equation: Stability analysis

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The noiseless Burgers equation in one spatial dimension is analyzed from the point of view of a diffusive evolution equation in terms of nonlinear soliton modes and linear diffusive modes. The transient evolution of the profile is interpreted as a gas of *right hand* solitons connected by ramp solutions with superposed linear diffusive modes. This picture is supported by a linear stability analysis of the soliton mode. The spectrum and phase shift of the diffusive modes are determined. In the presence of the soliton the diffusive modes develop a gap in the spectrum, and are phase shifted in accordance with Levinson's theorem. The spectrum also exhibits a zero-frequency translation or Goldstone mode associated with the broken translational symmetry. [S1063-651X(98)01002-2]

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I. INTRODUCTION

This is the first of a series of papers dedicated to an analysis of aspects of the Burgers equation in one spatial dimension. Driven with noise, this equation has the form

$$\frac{\partial u}{\partial t}=\nu\nabla^2 u+\lambda u\nabla u+\xi, \tag{1.1}$$

where u is the field in question, ν a damping constant or viscosity controlling the strength of the linear dissipative term, and λ a parameter characterizing the strength of the nonlinear mode coupling or convective term. The white noise ξ, characterizing the stochastic drive, is usually assumed to have a Gaussian distribution, and is correlated according to

$$\langle\xi(x,t)\xi(x',t')\rangle=K(x-x')\delta(t-t'), \tag{1.2}$$

where $K(x-x')$ accounts for the spatial correlations.

For $\lambda=-1$ and $\xi=0$, Eq. (1.1) was originally proposed by Burgers [1] as a model for irrotational or vorticity-free hydrodynamics in order to describe one dimensional turbulence [2–4]. In the noiseless case the large scale structure is dominated by shock waves and a detailed study of the transient decaying turbulence has been carried out [5–11].

The forced case for $\xi\neq 0$ was first considered by means of a dynamic renormalization group analysis in the context of long time tails in hydrodynamics [12]. Recently, the case of forced turbulence with random stirring at large length scales has received much attention, and has been treated by a variety of methods such as operator product expansions [13], instanton calculations [14], and replica methods [15]; see also Ref. [16]. Also, a more heuristic approach [17] to the stochastic dynamics of nucleation and coalescence of shocks has been advanced, recovering in a simple way some of the results from the operator product, instanton, and replica methods.

It turns out that the driven Burgers equation (1.1) for general λ in fact plays a paradigmatic role in modern nonequilibrium physics and describes a variety of apparently unrelated phenomena. In addition to modeling turbulence in one dimensional fluid flow, the equation has also been used to describe large scale pattern formation in astrophysics [18]; elastic lines in random media, e.g., vortices in superconductors; and growing interfaces [19].

In the context of stochastic growth the Burgers equation (1.1) governs the dynamics of the local slope $u=\nabla h$ of an interface and is driven by conserved short range noise, i.e., $\xi=\nabla\eta$, where η is correlated according to

$$\langle\eta(x,t)\eta(x',t')\rangle=\Delta\,\delta(x-x')\delta(t-t'). \tag{1.3}$$

The equation then takes the form

$$\frac{\partial u}{\partial t}=\nu\nabla^2 u+\lambda u\nabla u+\nabla\eta, \tag{1.4}$$

and the growing interface in terms of the height field $h=\int u\,dx$ is governed by the much studied Kardar-Parisi-Zhang equation [19,20]

$$\frac{\partial h}{\partial t}=\nu\nabla^2 h+\frac{\lambda}{2}(\nabla h)^2+\eta. \tag{1.5}$$

In a recent letter [21] we analyzed the noisy Burgers equation by a mapping of an equivalent discrete solid-on-solid model onto a spin-$\frac{1}{2}$ magnetic chain and a subsequent transition to a continuum field theory in the large spin limit. The approach provided a Hamiltonian description and gave insight into the strong coupling behavior of the noisy Burgers equation. In particular, the dynamic scaling exponent $z=\frac{3}{2}$ turns out to be determined by the dispersion law $E\propto p^z$ for the nonlinear soliton or shock wave solutions of the Hamiltonian field equations, replacing the noisy Burgers equation.

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We have recently generalized and unified the approach in Ref. [21] within the framework of the path integral formulation of the Martin-Siggia-Rose techniques [22]. The method is based on a weak noise saddle point approximation to the path integral, akin to the instanton methods advanced in the case of forced turbulence with stirring at large length scales mentioned above, and yields soliton or shock wave solutions of similar character as in the noiseless case. Details of this work will be presented in a forthcoming paper; for a brief account we refer to Ref. [23].

Since the dynamical aspects of the *noiseless* or *deterministic* Burgers equation, in particular its nonlinear soliton modes and superposed diffusive modes, play a decisive role in our analysis of the noisy case, we have found it convenient to break up our presentation, and in the present paper begin with a brief analysis of the noiseless case with special emphasis on the nonlinear soliton modes and their interaction with the linear diffusive spectrum. The present paper thus serves as a prelude, presenting and discussing some features which will persist in the presence of noise.

The noiseless Burgers equation has the form

$$\frac{\partial u}{\partial t}=\nu\nabla^2 u+\lambda u\nabla u. \tag{1.6}$$

Here ν is a damping constant controlling the strength of the linear dissipative term. The parameter λ characterizes the strength of the nonlinear mode coupling term. For $\lambda=-1$ Eq. (1.6) was introduced by Burgers [1] as a model for irrotational hydrodynamics. In the present context for general λ, we consider Eq. (1.6) as providing a description of the slope field $u=\nabla h$ for a growing interface governed by the noiseless or deterministic Kardar-Parisi-Zhang equation [20,24]

$$\frac{\partial h}{\partial t}=\nu\nabla^2 h+\frac{\lambda}{2}(\nabla h)^2. \tag{1.7}$$

It is an interesting property of the Burgers equation (1.6) that it is exactly soluble in the sense that the nonlinear Cole-Hopf transformation [25,26]

$$w=\exp\left[\frac{\lambda}{2\nu}\int^x u\,dx\right], \tag{1.8}$$

$$u=\frac{2\nu}{\lambda}\nabla\ \ln w \tag{1.9}$$

allows for an exact mapping onto the linear diffusion equation

$$\frac{\partial w}{\partial t}=\nu\nabla^2 w, \tag{1.10}$$

which can be simply analyzed. For given initial data $u(x,t=0)=u_0(x)$, we thus have

$$w(x,t)=\int G(x-x',t)\exp\left[\frac{\lambda}{2\nu}\int^{x'} u_0\,dx''\right], \tag{1.11}$$

where $G(x,t)$ is the Green's function for the diffusion equation (1.10)

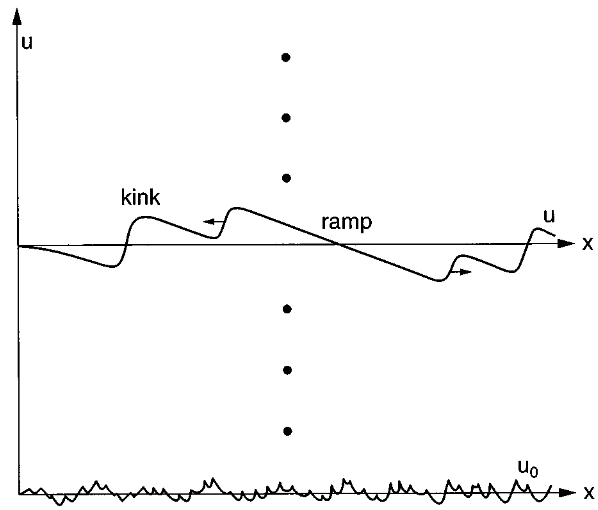

FIG. 1. Here we show in arbitrary units the transient evolution of the slope field u from an arbitrary initial configuration u_0. The transient morphology consists of propagating *right hand* solitons connected by ramps with superposed damped diffusive modes. Both the solitons and diffusive modes transport energy which is dissipated predominantly at the soliton positions. At long times the profile decays unless it is driven by currents at the boundaries, corresponding to a nonvanishing slope.

$$G(x,t)=[4\pi\nu t]^{-1/2}\exp\left[-\frac{x^2}{4\nu t}\right], \tag{1.12}$$

and u is given by Eq. (1.9).

The Cole-Hopf transformation [Eqs. (1.8) and (1.9)] thus permits a rather complete analysis of the Burgers equation. The relaxational dynamics of the equation is controlled by solitons connected by smooth regions; in the inviscid limit $\nu\rightarrow 0$ the solitons become shocks connected by ramps [2,4,7,8,20]. Although the nonlinear character of the equation prevents the application of a superposition principle, we can still from a qualitative point of view envisage that an initial configuration $u_0(x)$ ''contains'' a certain number of solitons or smoothed shocks. As time progresses the configuration passes through a transient regime dominated by a gas of propagating and coalescing solitons with superposed linear diffusive modes. At infinite times the configuration eventually decays, owing to the inviscid term in Eq. (1.6). This qualitative behavior is depicted in Fig. 1.

It is instructive to compare the nonlinear irreversible and dissipative Burgers equation (1.6) with the nonlinear reversible and dispersive evolution equations [27]: The equation of motion for the ϕ^4 field theory,

$$\frac{\partial^2\phi}{\partial t^2}=\nabla^2\phi+m^2\phi-\lambda\phi^3; \tag{1.13}$$

the complex nonlinear Schrödinger equation,

$$i\,\frac{\partial\psi}{\partial t}=\nabla^2\psi+\lambda|\psi|^2\psi; \tag{1.14}$$

and the sine-Gordon equation,

$$\frac{\partial^2 \phi}{\partial t^2} = \nabla^2 \phi + \lambda \sin\phi. \quad (1.15)$$

In addition to the linear dispersive modes obtained for $\lambda = 0$, the above evolution equations all support soliton solutions due to the dynamical balance between the linear dispersive term and the nonlinear term controlled by λ. An initial configuration thus again breaks up into a gas of moving solitons and linear modes. In the case of the sine-Gordon and nonlinear Schrödinger equations, the solitons preserve their identity under collisions owing to the complete integrability of these systems; this is not the case for the ϕ^4 field equation; here the solitons become deformed under collisions.

In the present paper we analyze the Burgers equation (1.6) from the point of view of a soliton-carrying dissipative evolution equation. The paper is organized in the following way. In Sec. II we summarize the general properties of the Burgers equation. In Sec. III we discuss in particular the soliton solution and comment on the morphology of a growing interface. In Sec. IV we present a linear stability analysis of the Burgers equation, and discuss the translation mode and the diffusive scattering modes. In Sec. V we summarize our results and present a conclusion.

II. GENERAL PROPERTIES

The Burgers equation (1.6) has the form of a nonlinear diffusive evolution equation with a linear diffusive term controlled by the damping or viscosity ν and a nonlinear mode coupling term characterized by λ. In the context of fluid motion the nonlinear term gives rise to convection as in the Navier-Stokes equation [1–3]; for an interface the term corresponds to a slope dependent growth [24].

Under time reversal $t \to -t$ and the transformation $u \to -u$, the equation is invariant provided $\nu \to -\nu$. This indicates that the linear diffusive and the nonlinear convective or growth terms play a completely different role. The diffusive term is intrinsically irreversible, whereas the growth term, corresponding to a mode coupling, gives rise to a cascade in wave number space and a genuine transient growth. The transformation $t \to -t$ is absorbed in $u \to -u$ or, alternatively, $\lambda \to -\lambda$, corresponding to a change of growth direction.

We also note that the equation is invariant under the parity transformation $x \to -x$, provided $u \to -u$. This feature is related to the presence of a single spatial derivative in the growth term, and implies that the equation only supports solitons or shocks with one parity, that is *right hand* solitons. We mention in passing that parity invariance is restored in the case of the noisy Burgers equation. This interesting aspect will be considered in a forthcoming paper.

The Burgers equation is also invariant under a more fundamental symmetry, namely, the Galilean symmetry group. In fact, boosting the equation to a moving frame with velocity λu_0, $x \to x - \lambda u_0 t$ it is easily seen by inspection, using $\partial/\partial t \to \partial/\partial t + \lambda u_0 \nabla$, that the equation remains invariant provided we shift the amplitude u by u_0, i.e., $u \to u + u_0$. We note that unlike the u^4 and sine-Gordon equations (1.13) and (1.15) which are invariant under a linear Lorentz transformation with no change in the field amplitude and the nonlinear Schrödinger equation (1.14) which is invariant under a Galilean transformation accompanied by a space and time dependent phase shift in the wave function, the Galilean transformation of the Burgers equation is nonlinear in the sense that the slope field is also shifted. Furthermore, the nonlinear coupling strength λ enters explicitly in the Galilean symmetry group.

In the absence of the nonlinear growth term for $\lambda = 0$, the Burgers equation (1.6) reduces to the linear diffusion equation (1.10) supporting linear diffusive modes $u \propto \exp(-i\omega t \pm ikx)$ with quadratic dispersion

$$\omega = -i\nu k^2. \quad (2.1)$$

An initial plane wave configuration thus decays with an envelope $\exp(-\nu k^2 t)$. More explicitly, defining the Laplace-Fourier transform

$$u(k,\omega) = \int dx\, dt\, \exp(i\omega t - ikx) u(x,t)\eta(t), \quad (2.2)$$

where $\eta(t)$ is the step function, i.e., $\eta(t) = 1$ for $t > 0$, $\eta(t) = 0$ for $t < 0$, and $\eta(0) = \frac{1}{2}$, and denoting the initial slope configuration by $u_0(k) = u(k, t=0)$, we have the solution

$$u(k,\omega) = \frac{u_0(k)}{-i\omega + \nu k^2}, \quad (2.3)$$

displaying a diffusive pole given by Eq. (2.1). For the temporal correlations we obtain, in particular,

$$\langle u(k,t) u(-k,t') \rangle = \langle u_0(k) u_0(-k) \rangle \exp[-(t+t')\nu k^2], \quad (2.4)$$

where $\langle \cdots \rangle$ denotes an average over the distribution of initial values.

On the other hand, in the inviscid limit for vanishing damping $\nu \to 0$ the Burgers equation (1.6) takes the form

$$\frac{\partial u}{\partial t} = \lambda u \nabla u, \quad (2.5)$$

which has an exact solution given by the implicit equation $u = F(x + \lambda u t)$, where F is an arbitrary profile. Since the propagation velocity λu thus increases with the amplitude u, it follows that an initial configuration $u_0 = F(x)$ breaks, and that shock waves are generated. From the form of the exact solutions it also follows that the shocks develop with right parity; i.e., a positive discontinuity in u. As mentioned above, this is consistent with the parity breaking properties of Eqs. (1.6) and (2.5). Searching for a static solution of the form $u = A + B\eta(x - x_0)$, we find, by insertion,

$$u(x) = |u_+| \eta(x - x_0), \quad (2.6)$$

with arbitrary amplitude $|u_+|$; a moving shock is then obtained by applying a Galilean boost, i.e., $x \to x - \lambda u_0 t$, $u \to u + u_0$, yielding shock solutions of Eq. (2.5). It is also easily seen that Eq. (2.5) supports ramp solutions of the form

$$u(x) = \text{const} - \frac{x}{\lambda t}. \quad (2.7)$$

The general picture that emerges in the inviscid limit $\nu \to 0$ is thus that an initial configuration $u_0 = F(x)$ breaks up into a series of *right hand* shocks connected by ramps; this picture is in fact substantiated by a steepest descent analysis of the Cole-Hopf transformation [Eqs. (1.8) and (1.9)] in the inviscid limit [8]. The time evolution is similar to the one depicted in Fig. 1, except that the solitons are sharp. Furthermore, in the absence of dissipation the transient regime extends to infinite times. In terms of the height field $h = \int u\, dx$, the morphology consists of cusps connected by convex parabolic segments [20,24].

In the presence of damping or viscosity this picture is not radically changed. The damping leads to an overall relaxation of the initial configuration where the energy, based on the hydrodynamical definition $\int u^2 dx$ (kinetic energy), is mainly dissipated in the shocks; in other words, the nonlinear mode coupling term gives rise to spatially confined *hot zones* for energy dissipation associated with the solitons.

III. SOLITON SOLUTION

Although the Burgers equation (1.6) admits an exact solution by means of the Cole-Hopf transformation [Eqs. (1.8) and (1.9)], we find it useful for our purposes to approach Eq. (1.6) as a diffusive nonlinear evolution equation and, following the corresponding analysis of Eqs. (1.13)–(1.15), to search for permanent profile soliton solutions [27]. Setting $u(x,t) = u(x - vt)$, where v is the propagation velocity, and using $\partial/\partial t = -v\nabla$, Eq. (1.6) can be integrated once

$$-vu = \nu\nabla u + \frac{\lambda}{2} u^2 + \text{const.} \tag{3.1}$$

Furthermore, imposing the boundary conditions $u = u_\pm$, $\nabla u \to 0$ for $x \to \pm\infty$, appropriate for a single soliton solution, and subtracting Eq. (3.1) for $x \to \pm\infty$, we obtain the soliton condition

$$u_+ + u_- = -\frac{2v}{\lambda}, \tag{3.2}$$

relating the propagation velocity v of the soliton to the boundary values $u_\pm$. We note that Eq. (3.2) is consistent with the fundamental nonlinear Galilean symmetry, being invariant under the transformation $v \to v + \lambda u_0$, $u_\pm \to u_\pm - u_0$. In terms of the boundary values $u_\pm$, we can also express Eq. (3.1) in the form

$$\nabla u = \frac{\lambda}{2\nu}(u_+ - u)(u - u_-), \tag{3.3}$$

which implies a positive slope of u between the boundary values $u_\pm$ and $u_+ > u_-$, corresponding to a *right hand* or positive parity soliton in accordance with the symmetry property discussed in Sec. II. In the static limit $v = 0$, Eq. (3.2) implies $u_+ = -u_-$, that is, a symmetric soliton, and by quadrature Eq. (3.3) yields the static soliton solution

$$u_0(x) = u_+ \tanh[k_s(x - x_0)], \tag{3.4}$$

$$k_s = \lambda u_+ / 2\nu. \tag{3.5}$$

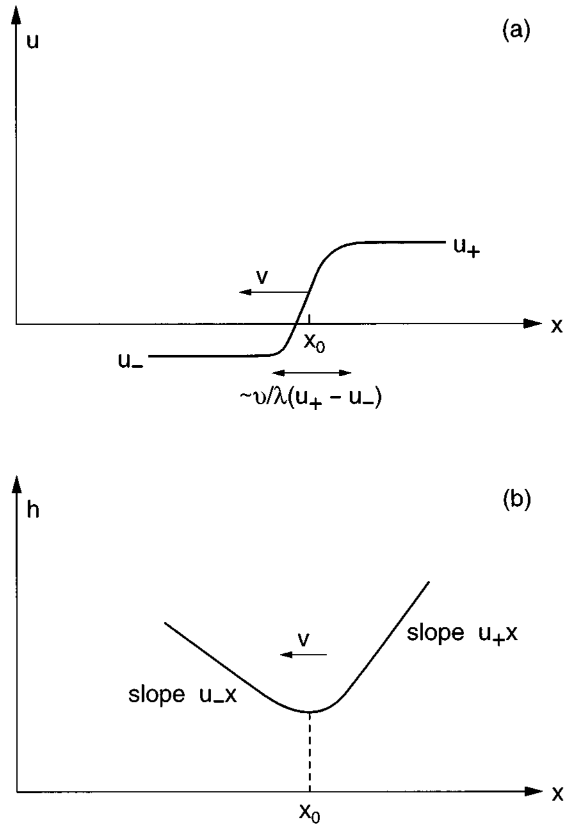

FIG. 2. In (a), we show a single moving soliton profile propagating to the left, and in (b) the corresponding smoothed cusp in the growth profile. This configuration is driven by currents at the boundaries, corresponding to nonvanishing $u_\pm$ and is persistent in time. In the plots we have chosen arbitrary units.

We have introduced the characteristic wave number k_s, setting the inverse length scale of the static soliton; x_0 denotes the center of mass position. The width of the soliton is of order $1/k_s$ and, unlike the ϕ^4 or sine-Gordon soliton, related to the amplitude u_+. In the inviscid limit $\nu \to 0$, the wave number $k_s \to \infty$ and the soliton approaches the shock discontinuity given by Eq. (2.6).

A moving soliton is obtained by applying a Galilean boost $x \to x - \lambda u_0 t$ and shifting the profile by u_0. In terms of the boundary values $u_\pm$, we thus obtain the propagating soliton solution

$$u_0(x,t) = \frac{u_+ + u_-}{2} + \frac{u_+ - u_-}{2} \tanh\left[\frac{\lambda}{4\nu}(u_+ - u_-)(x - vt - x_0)\right] \tag{3.6}$$

with velocity v given by Eq. (3.2). In Fig. 2 we depict a single soliton solution of the Burgers equation and the associated smoothed cusp profile for the height field of a growing interface.

IV. LINEAR STABILITY ANALYSIS

In order to investigate the properties of the linear diffusive modes in the presence of the nonlinear soliton mode we

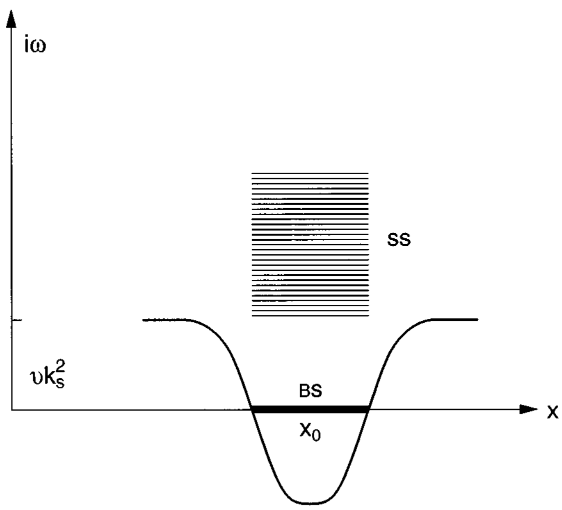

FIG. 3. Here we sketch in arbitrary units the reflectionless Bargman potential $\nu k_s^2(1-2/\cosh^2[k_s(x-x_0)])$. We also show the associated zero-frequency bound state (BS) or translation mode and the band of scattering states (SS) above the gap νk_s^2 in the spectrum.

here perform a linear stability analysis. Since a Galilean transformation allows for a boost to a finite propagation velocity, it is sufficient to consider the stability of the static soliton u_0 in Eq. (3.4). Setting $u=u_0+\delta u$ we obtain, by inserting into the Burgers equation (1.6) to linear order in δu the equation of motion for the fluctuations about the soliton profile,

$$\frac{\partial \delta u}{\partial t}=\nu\nabla^2\delta u+\lambda u_0\nabla\delta u+\lambda(\nabla u_0)\delta u. \tag{4.1}$$

Absorbing the first order derivative term by means of the transformation [28]

$$\delta u=\frac{\delta\tilde{u}}{\cosh\,[k_s(x-x_0)]}, \tag{4.2}$$

and searching for solutions with time dependence $\delta u \propto \exp(-i\omega t)$, we arrive at the linear eigenvalue problem

$$-i\omega\delta\tilde{u}=\nu\nabla^2\delta\tilde{u}-\nu k_s^2\left[1-\frac{2}{\cosh^2[k_s(x-x_0)]}\right]\delta\tilde{u}. \tag{4.3}$$

This equation has the same form as the one encountered in the linear stability analysis of the sine-Gordon soliton [27,29]. Interpreted as a stationary Schrödinger equation, Eq. (4.3) describes a particle with energy $i\omega$ and mass $1/2\nu$ in the exactly soluble Bargman potential $-2/\cosh^2 x$. The spectrum is well known, and consists of a single bound state for $\omega=0$ and a band of scattering states for $\omega=-i\nu(k^2+k_s^2)$ [29,30]. In Fig. 3 we sketch the potential and the band of scattering states.

A. Translation mode

The bound state solution for $\omega=0$ has the form $\delta\tilde{u}_{\mathrm{BS}} \propto 1/\cosh[k_s(x-x_0)]$ and using Eq. (4.2),

$$\delta u_{\mathrm{BS}}\propto\frac{1}{\cosh^2[k_s(x-x_0)]},\quad \omega=0. \tag{4.4}$$

This zero frequency mode has a particular significance for soliton-carrying systems. Since $\delta u_{\mathrm{BS}}\propto(du_0/dx)\delta x$, it is seen that the mode actually corresponds to an infinitesimal translation δx of the soliton without changing its shape. This mode thus restores the broken translational invariance, resulting from the choice of a particular center of mass position x_0 for the soliton—a so-called translation mode. Such Goldstone modes are quite generally associated with broken symmetries [27,29,31].

B. Diffusive scattering modes

In a similar way the band of scattering states has the explicit form

$$\delta u_{\mathrm{SS}}\propto\frac{\exp(ikx)}{\cosh[k_s(x-x_0)]}\,\frac{k+ik_s\tanh[k_s(x-x_0)]}{k-ik_s}, \tag{4.5}$$

with dispersion

$$\omega=-i\nu(k^2+k_s^2). \tag{4.6}$$

The modes form a continuum of spatially decaying diffusive states scattering off the soliton. We note that the dispersion law (4.6) for the diffusive spectrum, compared with Eq. (2.1) for the linear case, has developed a gap $\nu k_s^2=\lambda^2u_+^2/4\nu$ depending on the soliton amplitude. For $x\to\pm\infty$ we have $\delta u_{\mathrm{SS}}\propto\exp[ikx+i\delta(k)]\exp(-k_s|x|)$ and $\delta u_{\mathrm{SS}}\propto\exp(ikx)\exp(-k_s|x|)$, respectively, where the plane wave part is phase shifted by the amount

$$\delta(k)=2\tan^{-1}\left(\frac{k_s}{k}\right). \tag{4.7}$$

The transmission coefficient is thus given by $t(k)=\exp[i\delta(k)]$, i.e., $|t(k)|^2=1$, yielding the reflection coefficient $r(k)=0$, since $|r(k)|^2=1-|t(k)|^2=0$. Consequently, the soliton acts as a reflectionless transparent potential on the diffusive modes which only experience a phase shift. We also note that the bound state energy is given by the pole $k=ik_s$ in the S matrix $(k+ik_s)/(k-ik_s)$; the residue in Eq. (4.5) yields the bound state in Eq. (4.4). Furthermore, imposing periodic boundary conditions in a system of size N, we obtain the density of diffusive modes

$$\rho=\frac{N}{2\pi}+\frac{1}{2\pi}\frac{d\delta(k)}{dk}. \tag{4.8}$$

For the change of density of states owing to the presence of the soliton, $\Delta\rho=\rho-N/2\pi$, we have, inserting Eq. (4.7),

$$\Delta\rho=-\frac{1}{\pi}\frac{k_s}{k^2+k_s^2}. \tag{4.9}$$

We note that $\int\rho\,dk=N-1$ in accordance with Levinson's theorem, i.e., the band of diffusive modes is depleted by one mode corresponding to the zero frequency translation mode. In Fig. 4 we depict the phase shifted diffusive mode and the

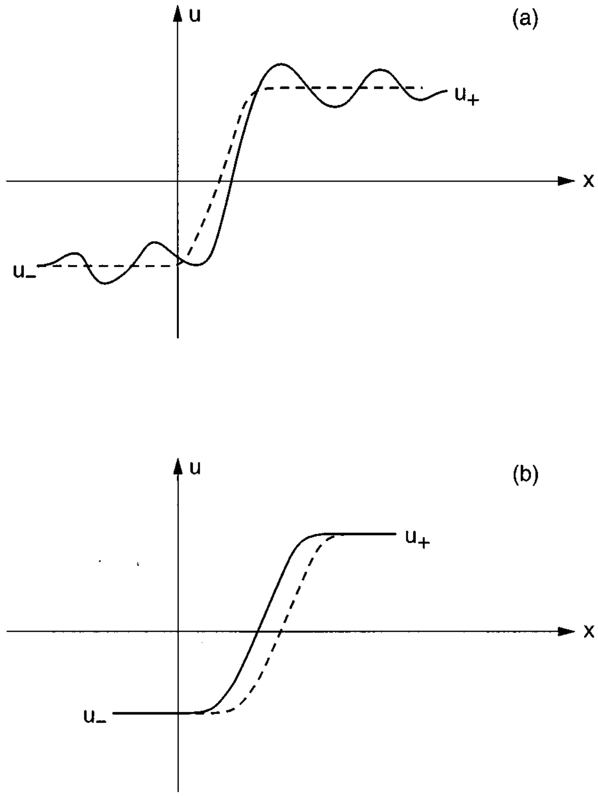

FIG. 4. In (a) we show the phase-shifted diffusive mode superposed on the static soliton, indicated by a dashed line. In (b) we depict the translation mode giving rise to a solid displacement of the soliton without changing its shape, thus lifting the broken translational invariance. The plots are in arbitrary units.

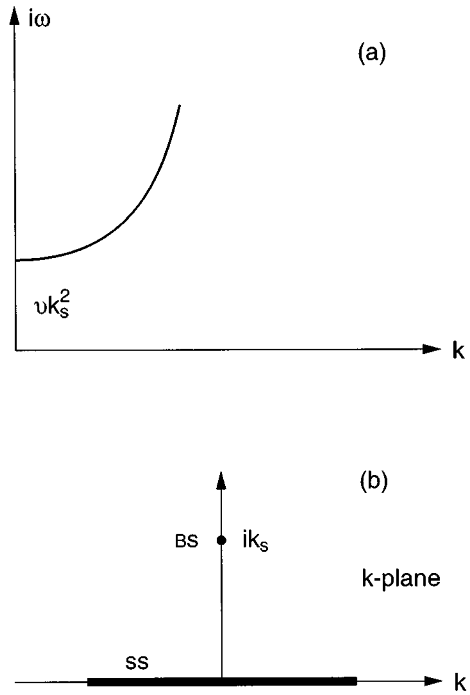

FIG. 5. In (a) we show the spectrum of diffusive modes in the presence of a soliton. Unlike the quadratic dispersion for the linear diffusion equation, the spectrum here exhibits a gap νk_s^2 depending on the soliton amplitude. In (b) we show the pole structure in the complex k plane. The real axis corresponds to the band of scattering states (SS). The pole in the S matrix on the imaginary axis corresponds to the bound state (BS) or translation mode. The plots are in arbitrary units.

translation mode. In Fig. 5 we show the diffusive spectrum (4.6) and the pole structure in the complex wave number plane.

Finally, regarding the stability of the soliton, we note that the linear diffusive mode is damped according to $\delta u_{\text{SS}\propto\exp}[-\Gamma(k)t]$ with a damping constant $\Gamma(k)=\nu(k^2+k_s^2)$. In the long wavelength limit $k\rightarrow 0$, $\Gamma(k)$ approaches a constant $\nu k_s^2=\lambda^2 u_+^2/4\nu$, i.e., the gap in the diffusive spectrum. This result is, however, not in conflict with the usual argument of a vanishing $\Gamma(k)$ for $k\rightarrow 0$, characteristic of a hydrodynamical diffusive mode [31]. The argument generally follows from the conservation of the local slope or velocity (momentum) implied by the structure of the Burgers equation (1.6), which can be written as a local conservation law

$$\frac{\partial u}{\partial t}=-\nabla j, \tag{4.10}$$

with a current density

$$j=-\nabla u-\frac{\lambda}{2}u^2. \tag{4.11}$$

In the long wavelength limit $k\rightarrow 0$, the conservation law (4.10) usually implies that the damping constant $\Gamma(k)\rightarrow 0$, i.e., an infinitely long lived mode. This follows from

$$\lim_{k\rightarrow 0}\frac{\partial u(k,t)}{\partial t}=\frac{d}{dt}\int u(x,t)dx=-\int \nabla j\ dx, \tag{4.12}$$

which vanishes *provided* that there are no currents on the boundaries $x=\pm N/2$. However, in the presence of a soliton there is an incoming current $j(\pm N/2)=-(\lambda/2u)_+^2$ and the mode decays with a finite lifetime $1/\nu k_s^2$ in the long wavelength limit, corresponding to a gap in the spectrum of $i\omega$.

V. SUMMARY AND CONCLUSION

In the present paper we conducted a study of the well-documented noiseless or deterministic Burgers equation, regarding it as a nonlinear diffusive evolution equation. Although the nonlinear Cole-Hopf mapping to a linear diffusive equation in principle allows for a rather complete analysis of the equation, we found it useful to emphasize the solitonic aspects, drawing on the parallel with other nonlinear equations such as the ϕ^4 and the sine-Gordon equations. Note, however, that unlike the ϕ^4 or sine-Gordon equations, where the soliton owes its stability to a balance between the dispersive effect of the linear term, tending to break up a wave packet construction, and the cascade effect in wave number space due to the nonlinear term, stabilizing a particular wave packet form (the soliton), the Burgers equation is intrinsically dissipative and an initial configuration will eventually decay due to dissipation *unless* energy is fed into the system. In this regard the Burgers soliton is a *dissipative* structure in that it owes its stability to the energy flux fed by the nonvanishing currents entering at the boundaries. The nonlinear term generating an *inverse cascade* in wave number space thus provides the energy transport to the center of the soliton, where the energy is dissipated and the soliton owes its stability to the interplay between the linear dissipative term and the nonlinear mode coupling term. Nevertheless, it is useful to consider the soliton as the fundamental *elementary excitation* in the Burgers equation determining the nonlinear nonlocal relaxational aspects. As follows from a steepest descent analysis in the inviscid limit $\nu \rightarrow 0$, an initial configuration evolves into a gas of propagating solitons connected by ramps. The present linear stability analysis then shows that the linear modes which for $\lambda=0$ dominate the relaxational by means of diffusion for $\lambda \neq 0$ become subdominant in the sense that they develop a gap in the $i\omega$ spectrum.

In a subsequent paper we consider the Burgers equation driven by spatially uniform stochastic noise rather that deterministic currents at the boundaries. We find that the solitonic aspects still determine the physics and that the soliton becomes a *bona fide* elementary excitation in the underlying field theory.

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[28] It is interesting to notice that a quick but fallacious *back of the envelope* argument applied to Eq. (4.1) in the limit $x \rightarrow \pm\infty$, where ∇u_0 vanishes, yields the complex dispersion law $\omega = -i\nu k^2 - \lambda u_0 k$. The correct result $\omega = -i\nu(k^2 + k_s^2)$, following from Eq. (4.3) is, of course, due to the influence of the term $\nabla u_0 \propto \cosh^{-2}[k_s(x-x_0)]$ at long distances.

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